satisfies

$$\kappa \gtrsim 4 \text{ GeV}^2 . \qquad (18)$$

Thus in order to see the scaling behavior for $\kappa = 1$ GeV^2 or larger we ought to have, for example, $\omega^{-1} \gtrsim 4$ GeV^{-1}, whereas for $\kappa \sim 0.5$ GeV^2 or larger we must have $\omega^{-1} \gtrsim 15$ GeV^{-1}. The numbers given here are only approximations. Nonetheless, they give fairly good estimates for the region of validity and show its sensitivity for small κ.

(3) It is particularly interesting to consider Eq. (16) for small κ, since we know that $\nu W_2(\kappa,\nu) = 0$ at $\kappa = 0$. From Eq. (17) we see that the scaling law should be valid even for very small κ, as long as ω^{-1} is large enough. Therefore, we should see $\nu W_2(\kappa,\nu)$ fall off as ω^{-1} becomes larger for any κ. The present experimental results do show the decreasing trend,[1] but it may be necessary to go to considerably higher ω^{-1} to see it more significantly. At any rate, as $\omega^{-1} \to \infty$ or $\omega \to 0$, we should have

$$W_2(\kappa,\nu) = F(\omega)/\nu \simeq \omega^\delta/\nu \simeq \kappa^\delta \nu^{-1-\delta} , \qquad (19)$$

where $\delta > 0$. This is indeed consistent with the Regge asymptotic behavior, viz.,

$$W_2(\kappa,\nu) \simeq \beta(\kappa) \nu^{\alpha(0)-2} \quad \text{as} \quad \nu \to \infty ,$$

where $\alpha(0) < 1$ is the $t = 0$ intercept of the leading Regge trajectory.

(4) Finally we note that we have neglected the contributions due to higher orders in γ throughout this paper. We remark that this is a meaningful approximation for $q_4/q_3 \lesssim \frac{1}{10}\pi$ or $\omega \lesssim \frac{1}{10}\pi\kappa^{1/2}$ which covers most of the region tested so far. However, the higher-order corrections become important in the region of small κ and large ω. We shall report on this along with other results in the future.

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Collins-Johnson Bootstrap Calculation of the $J=1$ $\pi\pi$ Scattering Amplitude

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Bounds are established for certain integrals involving $L(s)$, the left-hand cut contribution in $J=1$ $\pi\pi$ scattering. A theoretical calculation of $L(s)$ by Collins and Johnson is shown to violate the bounds seriously. The implications of this are discussed.

INTRODUCTION

In a series of papers, Collins and Johnson[1] have performed a calculation of the low-energy $\pi\pi$ scattering amplitudes, using the Frye-Warnock N/D equations.[2] The left-hand-cut contribution $L(s)$, which is needed as input to these equations, was calculated using a form of the "new strip approximation," which modifies and perhaps improves the single-ρ-exchange approximation. The calculation was apparently successful; in particular, it predicted a sufficiently narrow ρ meson without including a Castillejo-Dalitz-Dyson (CDD) parameter in the N/D equations.

In this paper the $J=1$ $\pi\pi$ partial wave is studied. A bound is established for a certain average over the *physical* quantity $L(s)$. It is shown that Collins and Johnson's calculated *theoretical* value L seriously violates this bound. The question then arises how it was possible for the N/D solution nevertheless to have the correct ρ resonance, and it is pointed out that a probable explanation is the absence of any CDD parameter in these equations. In other words, it is suggested that the correct ρ resonance was obtained only as a result of two canceling errors in the input—a too large left-hand-cut contribution and the absence of a CDD parameter.

It should be emphasized that the criticisms to be made concerning the Collins-Johnson calculation would probably apply equally to many other similar calculations. The point is, however, that, in contrast with most other calculations, the Collins-Johnson calculation gives a ρ meson with the correct mass and width, without a CDD parameter.

BOUNDS ON LEFT-HAND-CUT CONTRIBUTION

The Frye-Warnock N/D equations are most easily formulated using a function[3]

$$F(s) = (e^{i\delta} \sin\delta)/\eta^{-1}\rho , \qquad (1)$$

where

$$\rho(s) = \frac{s-s_0}{4}\left(\frac{s-s_0}{s}\right)^{1/2} \quad (s_0 = 4M_\pi^2) \qquad (2)$$

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[1] P. D. B. Collins and R. C. Johnson, Phys. Rev. **177**, 2472 (1969); **182**, 1755 (1969), **185**, 2020 (1969); The dot-dash curve in our Fig. 1 is taken from Fig. 5 of the second of these papers.

[2] G. Frye and R. L. Warnock, Phys. Rev. **123**, 1478 (1961).

[3] This is not quite the scattering amplitude $A(s) = (\eta e^{2i\delta} - 1)/2i\rho$, but its use allows a simple derivation of the Frye-Warnock equations and a simple definition of $L(s)$, though historically (see Ref. 2) $A(s)$ was used.

and δ and η are the phase shift and the inelasticity parameter. We *define* a "left-hand-cut contribution" L in terms of F by

$$\mathrm{Re}F(s)=L(s)+\frac{P}{\pi}\int_{s_0}^{20\,\mathrm{GeV}^2}ds'\frac{\mathrm{Im}F(s')}{s'-s}, \qquad (3)$$

the upper limit being the one used by Collins and Johnson.[1]

If δ and η were known up to sufficiently high energies (with a good accuracy for δ in the ρ resonance region), we could calculate an "experimental" value[4] for L and compare it with the calculated value of Collins and Johnson. Alternatively, with sufficiently sophisticated mathematics we could produce[5] rigorous bounds on weighted averages over L, using only Eq. (3) plus the unitarity limits

$$|\mathrm{Re}F(s)|\leqslant 1/2\rho(s)\,, \qquad (4)$$

$$0\leqslant \mathrm{Im}F(s)\leqslant 1/\rho(s)\,. \qquad (5)$$

Here a procedure is adopted which is intermediate between these two approaches.

Since the unitarity limit (5) diverges as $s\to s_0$, we consider the quantity

$$\left[L(s)+\frac{1}{\pi}\int_{s_0}^{s_1}ds'\frac{\mathrm{Im}F(s')}{s'-s}-\mathrm{Re}F(s)\right]=-\frac{P}{\pi}\int_{s_1}^{20\,\mathrm{GeV}^2}ds'\frac{\mathrm{Im}F(s')}{s'-s}, \qquad (6)$$

where s_1 is an arbitrary separation point and we only consider $s>s_1$. We then get rid of the principal-valued integral by averaging[6] over any interval $a<s<b$, with $a>s_1$ and $\frac{1}{2}(a+b)\leqslant 20$ GeV^2. If the square bracket is positive in this region (as it will be in our application) this gives the bound, using (4) and (5),

$$\int_a^b ds\left[L(s)-\frac{1}{\pi}\int_{s_0}^{s_1}ds'\frac{\mathrm{Im}F(s')}{s'-s}-\frac{1}{2\rho(s)}\right]\leqslant\frac{1}{\pi}\int_{s_1}^{\frac{1}{2}(a+b)}ds'\ln\left|\frac{b-s'}{s'-a}\right|\frac{1}{\rho(s')}. \qquad (7)$$

[The quantity $-(s'-s)^{-1}$ goes over to its average value $\ln|(b-s')/(s'-a)|$, which is positive for $s'<\frac{1}{2}(a+b)$.]

Present data do not permit a really accurate evaluation of the second term in the square bracket of (7). However, it will turn out to be enough to estimate its order of magnitude, and for this purpose we can use the narrow-width ρ-resonance approximation giving for a width of 120 MeV

$$-\frac{1}{\pi}\int_{s_0}^{s_1}ds'\frac{\mathrm{Im}F(s')}{s'-s}\simeq\frac{0.7}{s-M_\rho^2}\quad(s_1>M_\rho^2). \qquad (8)$$

For the s values we are going to consider, and for (at least) $s_1\sim M_\rho^2$, this approximation will not be wrong by an order of magnitude.

Now let us consider the Collins-Johnson calculated value for $L(s)$, which is shown in Fig. 1. We see that it exceeds $1/2\rho(s)$ for $s\gtrsim 2$ GeV^2. Also, according to the estimate (8) we can completely neglect the second term in the square bracket of (7), for (say) $s\gtrsim 4$ GeV^2.

In Table I, the inequality (8) is tested with this second term set equal to zero. It is seen to be violated, so we conclude that the Collins and Johnson's calculated value for $L(s)$ cannot be a good approximation to the physical value, over the range $2\lesssim s\lesssim 20$ GeV^2.

In the above, we have used physical information to show that the second term in the square bracket is small. Actually, it is likely that the Collins-Johnson value of $L(s)$ can be ruled out just by considering the unitarity limits (4) and (5), plus the dispersion relation (3).

To see this, we note that the integral

$$\int_{s_0}^{s_1}ds'\frac{\mathrm{Im}F(s')}{s'-s}$$

is over a relatively small range of s' values, in the sense that $(s'-s)^{-1}$ varies by only a few percent (for the values of s_1 and s which we need consider). Also, ImF is *positive*. These two facts mean that to a good approximation the integral will be of the form const$/(s-s_0)$ $\simeq$const$/s$.[7] It *cannot*, therefore, cancel the Collins-Johnson value for $L(s)$ to a good accuracy, over the whole range of s values involved; hence the inequality (7) [or its counterpart for the case where the square bracket in (6) is negative] is likely to be violated whatever form is assumed for ImF, provided this is positive.

That this conclusion is possible need cause no surprise, since it is already known[5] that $L(s)$ cannot be too big, as mentioned above.

If the Collins-Johnson value for $L(s)$ is indeed not permitted by the dispersion relation (3) and the unitarity bounds (4) and (5), the question arises

[4] This has been done for the πN case; see e.g., A. Donnachie, J. Hamilton, and A. T. Lea, Phys. Rev. **135**, B515 (1964).

[5] A. Martin, Nuovo Cimento **38**, 1326 (1965). (I am indebted to Dr. A. K. Common for this reference.) Rigorous bounds could also be obtained by the averaging method of the present paper, if one set $s_1=s_0$ but did the averaging with a weight function $\phi(s)$ such that $\int_a^b ds'\phi(s')/(s'-s_0)=0$ so as to avoid the difficulty that $1/\rho$ diverges as $s\to s_0$.

[6] The interchange of the orders of integration over s and s' to get the right-hand side of (7) may be justified as follows. First s is taken to vary over a pair of lines parallel to the real axis but finite distances $\pm\epsilon$ away, so that no principal-valued integral is necessary. The interchange of the orders of integration for finite ϵ is then immediate, because the integrand is bounded. Finally, the interchange of the limit $\epsilon\to 0$ with the integration over s' is permissible because of Lebesgue's dominated-convergence theorem. I am indebted to Dr. D. Atkinson for demanding that I provide a proof of (7).

[7] The positive definiteness of ImF allows one to calculate a precise envelope around the curve const$/s$, within which the integral must lie; see A. P. Balachandran, Ann. Phys. (N. Y.) **30**, 476 (1964).

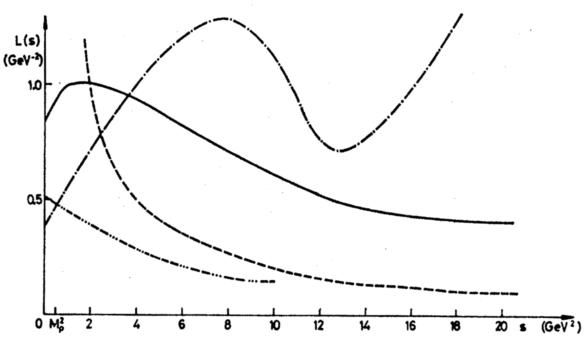

FIG. 1. Curves shown are: – – –, $1/2\rho(s)$; –·–, the Collins-Johnson $L(s)$; ——, unmodified ρ exchange for $L(S)$, with ρ width 120 MeV; –····–, a Veneziano model prediction for $L(s)$ [S. Humble (unpublished)], with ρ width 120 MeV. Note that the Collins-Johnson value is even bigger than unmodified ρ exchange, whereas the Veneziano-model prediction is smaller and would certainly not violate any inequality of the kind considered in this paper.

how an N/D solution could be obtained using this value for $L(s)$, since the N/D solution is automatically unitary. The answer, of course, must be that the dispersion relation (3) is violated by the N/D solution. This will happen if the solution has poles in addition to right- and left-hand cuts. It is known[8] that with the elementary ρ-exchange approximation for L, an infinite number of such poles arises in the infinite-cutoff limit.

CONSEQUENCES OF VIOLATION OF BOUND

Since the Collins-Johnson value for L is unphysical, one has to ask why they obtained the correct ρ resonance. Obviously there are two possibilities, (i) the calculation is insensitive to large changes in $L(s)$, or (ii) an additional, canceling, nonphysical feature occurs in the calculation.

Regarding (i), experience with N/D calculations[9] suggests that it is not the case; also it would be undesirable for the Collins-Johnson calculation, where a bootstrap requirement is being imposed to determine L by varying it until the output satisfies certain criteria.

Regarding (ii), it is noteworthy that a serious candidate for the second unphysical feature is provided by the fact that a CDD parameter is likely to be necessary in the physical N/D equations. As explained, for example, in Ref. 9, this will be necessary unless the phase shift $\delta(s)$ falls quickly to zero above the ρ resonance. It is not known whether the $\pi\pi$ phase shift falls like this, but the resonant πN phase shifts—most strikingly the P_{33}—certainly do not.

[8] A. P. Contogouris and A. Martin, Nuovo Cimento **49A**, 61 (1967).

[9] See, e.g., R. C. Devenish, J. C. Eilbeck, and D. H. Lyth, University of Lancaster Report, 1970 (unpublished).

TABLE I. Violation of inequality (7). The inequality (7) is shown to be violated, with $b=20$ GeV^2 and with a and s_1 as shown. As explained in the text, the second term in the square bracket of (7) is set equal to zero.

s_1 (GeV^2)	$a=8$ GeV^2 L.h.s.$\leq$	R.h.s.	$a=2$ GeV^2 L.h.s.$\leq$	R.h.s.
2	8	3.6	11.5	4.0
1	8	4.8	11.5	6.4
0.5	8	5.2	11.5	9.6

CONCLUSION

The *firm conclusion* is that the Collins-Johnson calculation of $L(s)$ has given a result bigger than the physical one. An argument is also given which suggests that the calculated N/D solution has important spurious poles. It is emphasized that a CDD parameter is likely to be needed in the physical solution; if this is so, the success of the calculation (which had no CDD parameter) was possible only because an unphysical $L(s)$ was used as input.

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